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Compact Formal Analysis of Beam Columns

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A RECENT note[1] by Urry considers the use of what he calls Macauley's brackets in the treatment of beam-column problems. It is the purpose of the present note to generalize this treatment. The notation we will use

$$\{x - a\}^n = \left.\begin{matrix} (x-a)^n \text{ if } x \geqq a \\ = 0 \quad\quad \text{if } x < a \end{matrix}\right\} n \geqq 0$$

is that of Brock and Newton[2]; cf. a recent note by Weissenburger which discusses history and compares notations. We will also use the following additional notation:

$$\{x - a\} = \{x - a\}^1$$

That is, the unit exponent will not be exhibited. Furthermore, suppose that $f(z)$ is a given function of the variable z; we will use the notation

$$\begin{aligned} f[\alpha\{x - a\}] &= f[\alpha(x - a)] \text{ if } x \geqq a \\ &= f(0) \quad\quad \text{if } x < a \end{aligned}$$

Thus, for example,

$$\begin{aligned} \cos[\alpha\{x - a\}] &= \cos[\alpha(x - a)] \text{ if } x \geqq a \\ &= 1 \quad\quad \text{if } x < a \end{aligned}$$

In the analysis, we first consider the following functions that result from truncating the Maclaurin series for sine and cosine:

$$\text{tru}_n z = \frac{z^n}{\Gamma(n+1)} - \frac{z^{n+2}}{\Gamma(n+3)} + \frac{z^{n+4}}{\Gamma(n+5)} - \cdots$$

In most practical applications, n is a positive integer so that the gamma functions may be replaced by factorials, viz.,

$$\Gamma(n + 1) = n! \text{ (for integer } n)$$

However, the more general form may be useful in certain cases. It is easily established that

$$d(\text{tru}_n z)/dz = \text{tru}_{n-1} z \qquad n \neq 0$$

$$d^2(\text{tru}_n z)/dz^2 = \text{tru}_{n-2} z = [z^{n-2}/\Gamma(n-1)] - \text{tru}_n z \qquad n \neq 0,1$$

The first several tru-functions for integer index are:

$$\begin{aligned} \text{tru}_0 z &= \cos z \\ \text{tru}_1 z &= \sin z \\ \text{tru}_2 z &= 1 - \cos z \\ \text{tru}_3 z &= z - \sin z \\ \text{tru}_4 z &= \cos z - (1 - z^2/2!) \\ \text{tru}_5 z &= \sin z - (z - z^3/3!), \ldots \end{aligned}$$

Now consider the problem of a laterally and axially loaded beam of uniform flexural rigidity EI and length L. Let M^* denote the bending moment that would be acting if the axial load were absent. One may write

$$M^* = \Sigma c_k\{x - a_k\}^{n_k}$$

Some of the coefficients c_k may represent redundant reactions, constraining moments, etc., in the case of statically indeterminate beams. Also, there may exist relations involving M^* and/or its derivative; for example, $M^* = 0$ at a hinged support. In the case of statically indeterminate beam columns, there will be additional conditions on deflection and/or slope which, later, will provide sufficient conditions for the determination of all the c_k and the integration constants A and B introduced in what follows.

Because of the additional compressive axial load P, there is an additional contribution, $-Py$, to the bending moment; here y denotes the deflection. Thus, we are concerned with the differential relation

$$(d^2y/dx^2) + \alpha^2 y = M^*/EI \qquad \alpha^2 = P/EI$$

It may be easily verified by direct substitution that the function

$$y = A\cos(\alpha x) + B\sin(\alpha x) + \Sigma \frac{c_k(n_k)!}{P\alpha^{n_k}} \text{tru}_{n_k+2}[\alpha\{x - a_k\}]$$

satisfies this differential equation if M^* has the form given earlier. This may be considered to be the formal generalization of the use of the curly-bracket symbol (the name we prefer to use in order to prevent confusion of nomenclature; cf. Weissenburger[3]) in the beam-column problem. If the axial load is tensile, P is to be considered as negative and α becomes imaginary; however, it should be clear how algebraic signs may be manipulated and hyperbolic functions may be introduced so as to obtain convenient expressions.

It may be of interest to consider in somewhat greater detail the important case of the pin-ended statically determinate beam column. In the general case, the moment M^* due to lateral loads alone is

$$M^* = \Sigma c_k[xb_k^{n_k} - L\{x - a_k\}^{n_k}]$$

where the corresponding loading starts at $x = a_k = L - b_k$, the coefficient c_k has the dimension: (force $\times$ length$^{-n_k}$), and the exponents have the following significance: 0 $\sim$ concentrated clockwise moment, 1 $\sim$ concentrated downward load, 2 $\sim$ uniformly distributed downward load, and, for n greater than 2, n $\sim$ downward distributed load of the form $w_0\{x - a_k\}^{n-2}$. Furthermore, the condition of zero deflection at the ends permits evaluating the constants A and B, and the complete solution may be written

$$y = \Sigma \frac{Lc_k}{P}\left[b_k^{n_k}\left(\frac{x}{L} - \frac{\sin\alpha x}{\sin\alpha L}\right) - \frac{(n_k)!}{\alpha^{n_k}}\left(\text{tru}_{n_k+2}[\alpha\{x - a_k\}] - \frac{\sin\alpha x}{\sin\alpha L}\text{tru}_{n_k+2}(\alpha b_k)\right)\right]$$

As an example, if there is a single downward concentrated load F at $x = a$, we have

$$y = \frac{Fbx}{PL} - \frac{F\sin(\alpha b)\cdot\sin(\alpha x)}{P\alpha\sin(\alpha L)} - \left(\frac{F}{P\alpha}\right)(\alpha\{x - a\} - \sin[\alpha\{x-a\}])$$

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In a similar manner, the complete solution for the clamped-clamped beam column may be written

$$y = \Sigma \frac{Lc_k(n_k)!}{P\Delta\alpha^{n_k}} (\beta_k \operatorname{tru}_2(\alpha x) + \gamma_k \operatorname{tru}_3(\alpha x) - \Delta \operatorname{tru}_{n_k+2}[\alpha\{x - a_k\}])$$

where

$$\beta_k = \operatorname{tru}_{n_k+2}(\alpha b_k)\cdot \operatorname{tru}_2(\alpha L) - \operatorname{tru}_{n_k+1}(\alpha b_k)\cdot \operatorname{tru}_3(\alpha L)$$
$$\gamma_k = \operatorname{tru}_{n_k+1}(\alpha b_k)\cdot \operatorname{tru}_2(\alpha L) - \operatorname{tru}_{n_k+2}(\alpha b_k)\cdot \operatorname{tru}_1(\alpha L)$$
$$\Delta = \operatorname{tru}_2(\alpha L)\cdot \operatorname{tru}_2(\alpha L) - \operatorname{tru}_1(\alpha L)\cdot \operatorname{tru}_3(\alpha L) = 2 - 2\cos(\alpha L) - \alpha L \sin(\alpha L)$$

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Natural Frequencies of Meridional Vibration in Thin Conical Shells

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Nomenclature

u_S, u_θ, u_r	= meridional, tangential, and normal displacement, in.
$\epsilon_S, \epsilon_\theta, \epsilon_{s\theta}$	= meridional, tangential, and shear strain, in./in.
$\sigma_s, \sigma_\theta, \sigma_{s\theta}$	= meridional, tangential, and shear stress, psi
G	= shear modulus, psi
E	= tensile modulus, psi
ν	= Poisson's ratio
ρ	= mass density, #sec²/in.⁴
ω	= circular frequency, rad/sec
α	= cone half angle, deg
t	= cone wall thickness, in.
L^*	= length of cone along meridian, in.
J_n, Y_n	= Bessel functions of first and second kind
r_0, r_1	= cone radii, in.
L	= length of cone along axis, in.
a	= $(E/\rho)^{1/2}$ = velocity of sound in the material, in./sec
b	= $r_0 L^*/r_1 - r_0$, in.
β	= r_0/r_1
Ω	= $\omega L^*/a$ = frequency parameter, rad/sec
r_S	= $r_0 + (r_1 - r_0/L^*)S$, in.; see Fig. 1
A_S	= $2\pi r_s t$ = cone cross-sectional area, in.²

Introduction

ACOUSTICAL instability, sometimes present in solid propellant rocket motors, may result in the generation of longitudinal vibratory forces of sufficient magnitude to threaten structural integrity of the various motor components. A knowledge of the resonant frequency characteristics of these components in conjunction with predicted acoustic frequencies is therefore of primary importance to the motor designer.

Reasonably accurate estimates can be obtained for resonant frequencies of axial vibration in bars of constant cross section, neglecting lateral inertia effects, using methods readily found in existing literature.[1] In many instances, the motor components to be investigated do not satisfy this condition of constant cross-sectional area. For example, most motor nozzles and some types of interstage structures fall into a category more readily represented by a thin conical shell.

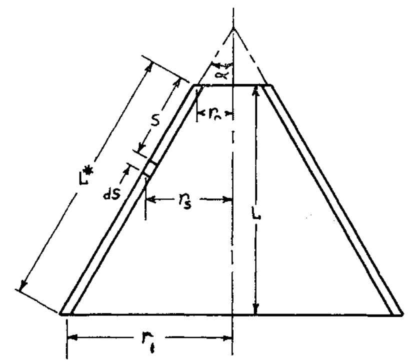

Fig. 1 Conical shell geometry.

The work presented here deals with the prediction of natural frequencies of meridional vibration in conical shells of constant wall thickness. The results can be considered to be an extension of the approximate theory for uniform bars just described since lateral inertia effects have been neglected in the formulation.

Discussion

Consider the thin-walled conical shell of constant wall thickness shown in Fig. 1. The general stress-strain and strain-displacement relations for a thin conical shell are[2]:

$$\sigma_S = \frac{E}{1-\nu^2}(\epsilon_S + \nu\epsilon_\theta) \tag{1}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}(\epsilon_\theta + \nu\epsilon_S) \tag{2}$$

$$\sigma_{S\theta} = G\epsilon_{S\theta} \tag{3}$$

$$\epsilon_S = \partial u_S/\partial S \tag{4}$$

$$\epsilon_\theta = \frac{1}{r_S}\frac{\partial u_\theta}{\partial \theta} + \frac{u_n \cos\alpha - u_S \sin\alpha}{r_S} \tag{5}$$

$$\epsilon_{S\theta} = \frac{1}{2}\left[\frac{\partial u_\theta}{\partial S} + \frac{1}{r_S}\frac{\partial u_S}{\partial \theta} + \frac{u_\theta \sin\alpha}{r_S}\right] \tag{6}$$

The assumption is now made that, during meridional vibration, the cross sections of the cone remain plane, and particles in these cross sections perform only motions in the meridional direction. This results in a system of membrane loading in which

$$\sigma_\theta = \sigma_{S\theta} = \epsilon_{S\theta} = 0$$

Upon substitution of these conditions into Eqs. (1–6) we obtain

$$\sigma_S = E\epsilon_S = E(\partial u_S/\partial S) \tag{7}$$

The meridional force at S is given as

$$F_S = A_S\sigma_S = A_S E(\partial u_S/\partial S) \tag{8}$$

The change in force across the element (ds) is given as

$$dF_S = EdS\, 2\pi t\left[r_S\frac{\partial^2 u_S}{\partial S^2} + \frac{r_1 - r_0}{L^*}\frac{\partial u_S}{\partial S}\right] \tag{9}$$

The meridional inertia force of the element at s is given as

$$F_I = -M_S\ddot{u}_S = -2\pi r_S t\rho dS\ddot{u}_S \tag{10}$$

Upon application of D'Alembert's principle, the following differential equation of motion results:

$$\frac{\partial^2 u_S}{\partial S^2} + \frac{1}{b+S}\frac{\partial u_S}{\partial S} = \frac{1}{a^2}\frac{\partial u_S}{\partial t} \tag{11}$$

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